

EQUIVALENCE OF LAWS AND NULL CONTROLLABILITY FOR SPDES DRIVEN BY A FRACTIONAL BROWNIAN MOTION

BOHDAN MASLOWSKI AND JAN VAN NEERVEN

ABSTRACT. We obtain necessary and sufficient conditions for equivalence of law for linear stochastic evolution equations driven by a general Gaussian noise by identifying the suitable space of controls for the corresponding deterministic control problem. This result is applied to semilinear (reaction-diffusion) equations driven by a fractional Brownian motion. We establish the equivalence of continuous dependence of laws of solutions to semilinear equations on the initial datum in the topology of pointwise convergence of measures and null controllability for the corresponding deterministic control problem.

1. INTRODUCTION

The equivalence (mutual absolute continuity) of probability distributions of solutions to infinite-dimensional stochastic equations has been extensively studied for equations driven by Wiener process and is of importance in the investigation of large-time behaviour and ergodicity of solutions (see e.g. [14, 15, 21] for early works on stochastic reaction-diffusion equations, the book [6], and the references therein). Indeed, by Doob's theorem and its improvements [24, 25] it enables proofs of strong ergodicity and mixing of the Markov semigroups.

The first results in this direction for stochastic linear equations appeared in the pioneering monograph by Da Prato and Zabczyk [5] (cf. also the earlier paper by J. Zabczyk [27]), where the following statement may be found. Consider the equation

$$(1.1) \quad \begin{cases} dX(t) = AX(t) dt + B dW, \\ X(0) = x, \end{cases}$$

in a real separable Hilbert space E , where A is a densely defined linear operator on E generating a strongly continuous semigroup, B is bounded on E and W is a standard cylindrical Wiener process in E . Assuming the existence of a unique E -valued mild solution, for a given $T > 0$ the probability laws of the solutions are equivalent for different values of the initial datum $x \in E$ if and only if the corresponding deterministic controlled system

$$(1.2) \quad \begin{cases} y' = Ay + Bu, \\ y(0) = x, \end{cases}$$

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is exactly null controllable at T with controls from the set $L^2(0, T; E)$, i.e., for each $x \in E$ there exists a control $u \in L^2(0, T; E)$ such that $y(T) = 0$. This made it possible to utilize the well-developed deterministic controllability theory for proving equivalence of laws for the stochastic equation (1.1).

In the non-Markovian case the problem of equivalence of laws was addressed in the paper [7] in the so-called diagonal case (when the operators A and B commute) with a fractional Brownian motion as the driving process. Here we extend the result of this paper by providing necessary and sufficient conditions for the equivalence of laws (Example 3.4). This toy model shows the interesting (but natural) feature that smoother paths of the driving process (i.e. a bigger Hurst parameter) require more restrictive conditions for equivalence of laws (which, of course, is just the converse with respect to regularity problems).

In Section 2 the first part of the paper we obtain an extension of the above-mentioned result from [5, 27] to the case of general Gaussian noise. The state space E is a general Banach space and the objective is to identify the suitable space of controls such that the above-described relation between equivalence of laws for (1.1) and null controllability for (1.2) holds true.

In Sections 3 and 4 these results are applied to semilinear (reaction-diffusion) equations driven by a fractional Brownian motion; this part is based on the density formula established in [9]. The main result here is the equivalence of continuous dependence of laws of solutions to semilinear equations on the initial datum in topology of pointwise convergence of measures and null controllability for the corresponding control problem. As in the Markov case, we call this property the *strong Feller property*. The basic idea of the proof of this result is due to Masłowski and Seidler [16]. In these results, for simplicity we assume that the space E is Hilbert. Using the ideas of [3], the results can be extended to (type 2) Banach spaces, but our examples would not be improved much by this extension.

2. NULL CONTROLLABILITY AND EQUIVALENT OF LAWS

Let \mathcal{H} be a real Hilbert space with inner product $[\cdot, \cdot]_{\mathcal{H}}$, and let W be an \mathcal{H} -isonormal process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By definition, this means that W is a bounded linear mapping from \mathcal{H} into $L^2(\Omega)$ such that the following two conditions are satisfied:

- (1) for all $h \in \mathcal{H}$, the random variable $W(h)$ is centred Gaussian;
- (2) for all $h_1, h_2 \in \mathcal{H}$,

$$\mathbb{E}W(h_1)W(h_2) = [h_1, h_2]_{\mathcal{H}}.$$

Let E be a real Banach space. We denote by $\gamma(\mathcal{H}, E)$ the completion of the algebraic tensor product $\mathcal{H} \otimes E$ with respect to the norm

$$\left\| \sum_{n=1}^N h_n \otimes x_n \right\|_{\gamma(\mathcal{H}, E)}^2 = \mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|^2,$$

where $(h_n)_{n=1}^N$ is an orthonormal sequence in \mathcal{H} , $(x_n)_{n=1}^N$ is a sequence in E , and $(\gamma_n)_{n=1}^N$ is Gaussian sequence, i.e., a sequence of independent standard real-valued Gaussian random variables. The natural inclusion mapping $\mathcal{H} \otimes E \hookrightarrow \mathcal{L}(\mathcal{H}, E)$ extends to a continuous inclusion mapping $\gamma(\mathcal{H}, E) \hookrightarrow \mathcal{L}(\mathcal{H}, E)$; this allows us to identify elements of $\gamma(\mathcal{H}, E)$ with bounded operators acting from \mathcal{H} to E . The

operators belonging to $\gamma(\mathcal{H}, E)$ are called the γ -radonifying operators from \mathcal{H} to E .

The mapping $W : \mathcal{H} \rightarrow L^2(\Omega)$ extends to an isometry, also denoted by W , from $\gamma(\mathcal{H}, E)$ into $L^2(\Omega; E)$ by putting

$$W(h \otimes x) := W(h) \otimes x.$$

Moreover, $\langle W(R), x^* \rangle = W(R^* x^*)$ for all $R \in \gamma(\mathcal{H}, E)$ and $x^* \in E^*$. This extension can be viewed as an abstract stochastic integral, as can be seen by considering the case $\mathcal{H} = L^2 := L^2(0, T)$ (see [18, 19]):

Example 2.1. If W is an L^2 -isonormal process, then $B_t := W(\mathbf{1}_{(0,t]})$ defines a standard real-valued Brownian motion $(B_t)_{t \in [0, T]}$ and for all $h \in H$ we have

$$W(h) = \int_0^T h(t) dB_t.$$

Moreover, the extension $W : \gamma(L^2, E) \rightarrow L^2(\Omega; E)$ coincides with the E -valued stochastic integral introduced in [19]. Conversely, if $(B_t)_{t \in [0, T]}$ is a standard real-valued Brownian motion $(B_t)_{t \in [0, T]}$, then above identity defines an L^2 -isonormal process W .

Thus, L^2 -isonormal process are in one-to-one correspondence with standard real-valued Brownian motions.

Example 2.2. Let H be a real Hilbert space and let $L^2(H) := L^2(0, T; H)$. An H -cylindrical Brownian motion is an $L^2(H)$ -isonormal process W . For any real Banach space E , the associated stochastic integral $W : L^2(H) \rightarrow L^2(\Omega)$ extends to an isometry W from $\gamma(L^2(H), E)$ into $L^2(\Omega; E)$ (see [19]).

In what follows it will be useful to have an explicit description of the reproducing kernel Hilbert space (RKHS) associated with the (centred Gaussian) random variables $W(R)$ associated with γ -radonifying operators R :

Proposition 2.3. *Let W be an \mathcal{H} -isonormal process and let $R \in \gamma(\mathcal{H}, E)$ be a γ -radonifying operator. The reproducing kernel Hilbert space \mathcal{G}_R associated with $W(R)$ equals the range of R .*

Proof. Let $Q_R \in \mathcal{L}(E^*, E)$ denote the covariance operator of $W(R)$. For all $x^* \in E^*$ we have

$$\langle Q_R x^*, x^* \rangle = \mathbb{E} |\langle W(R), x^* \rangle|^2 = \mathbb{E} |W(R^* x^*)|^2 = \|R^* x^*\|_{\mathcal{H}}^2 = \langle RR^* x^*, x^* \rangle.$$

Hence $RR^* = Q_R = i_R i_R^*$, where $i_R : \mathcal{G}_R \hookrightarrow E$ is the canonical embedding. It follows that R maps $\text{ran}(R^*)$ into $\text{ran}(i_R) = \mathcal{G}_R$. As an element of \mathcal{G}_R , $RR^* x^*$ equals $i_R^* x^*$, so

$$\|RR^* x^*\|_{\mathcal{G}_R}^2 = \|i_R^* x^*\|_{\mathcal{G}_R}^2 = \langle Q_R x^*, x^* \rangle = \|R^* x^*\|^2.$$

It follows that R extends to an isometry from $\overline{\text{Range}(R^*)}^{\mathcal{H}}$ onto $\overline{\text{Range}(i_R^*)}^{\mathcal{G}_R} = \mathcal{G}_R$. Finally, if $h \perp \overline{\text{Range}(R^*)}^{\mathcal{H}}$ then $Rh = 0$. \square

From this point on, we fix a Hilbert space H and a finite time horizon $0 < T < \infty$, and write $L^2(H) := L^2(0, T; H)$. We fix a function $\Phi : (0, T) \rightarrow \mathcal{L}(H, E)$ which has the property that the adjoint orbits $t \mapsto \Phi^*(t)x^*$ belong to $L^2(H)$. Here,

$\Phi^*(t) = (\Phi(t))^* : E^* \rightarrow H$ denotes the adjoint of $\Phi(t) : H \rightarrow E$ defined via the Riesz representation theorem.

In order to discuss stochastic integrability of Φ with respect to the \mathcal{H} -isonormal process W , we need to connect the spaces \mathcal{H} and $L^2(H)$. This will be done in the next two subsections, where we consider the situations where we have continuous dense embeddings $\mathcal{H} \hookrightarrow L^2(H)$ and $L^2(H) \hookrightarrow \mathcal{H}$, respectively. These embeddings can be interpreted as saying that W is ‘rougher’, respectively ‘smoother’, than H -cylindrical motions. This rough noise case is slightly subtler to deal with and will therefore be presented in detail; the smooth noise case proceeds entirely analogous, with some slight simplifications.

The basic examples we have in mind are provided by H -cylindrical (classical, Liouville) fractional Brownian motions; see Section 3.

2.1. The rough noise case. In this subsection we make the following assumption.

Assumption 2.4. *The space \mathcal{H} is continuously and densely included in $L^2(H)$.*

We then have continuous and dense embeddings $\mathcal{H} \hookrightarrow L^2(H) \hookrightarrow \mathcal{H}^*$, where \mathcal{H}^* denotes the dual of \mathcal{H} relative to the $L^2(H)$ -duality. Thus, for all $h \in \mathcal{H}$ and $f \in L^2(H)$ we have

$$\langle h, f \rangle_{\langle \mathcal{H}, \mathcal{H}^* \rangle} = \int_0^T h(t) f(t) dt.$$

For each $h \in \mathcal{H}$ we define the element $\phi_h \in \mathcal{H}^*$ by

$$\langle g, \phi_h \rangle_{\langle \mathcal{H}, \mathcal{H}^* \rangle} = [g, h]_{\mathcal{H}}.$$

By the Riesz representation theorem, the correspondence $h \leftrightarrow \phi_h$ sets up a bijective correspondence between \mathcal{H} and \mathcal{H}^* .

For a mapping S from a Banach space X into \mathcal{H} we denote by $S^* : \mathcal{H}^* \rightarrow X^*$ the adjoint, so that for all $x \in X$ and $h^* \in \mathcal{H}^*$ we have

$$\langle x, S^* h^* \rangle = \langle Sx, h \rangle_{\langle \mathcal{H}, \mathcal{H}^* \rangle}.$$

Definition 2.5. *The function $\Phi : (0, T) \rightarrow \mathcal{L}(H, E)$ is said to be stochastically integrable with respect to the \mathcal{H} -isonormal process W if $t \mapsto \Phi^*(t)x^*$ belongs to \mathcal{H} for all $x^* \in E^*$ and there exists an operator $R_\Phi \in \gamma(\mathcal{H}, E)$ such that $R_\Phi^* x^* = \Phi^* x^*$ for all $x^* \in E^*$. The random variable $W(R_\Phi)$ is called the stochastic integral of Φ with respect to W , notation*

$$\int_0^T \Phi dW = W(R_\Phi).$$

Here, $R_\Phi^* : E^* \rightarrow \mathcal{H}$ denotes the adjoint of $R_\Phi : \mathcal{H} \rightarrow E$ defined via the Riesz representation theorem.

Proposition 2.6. *Let Assumption 2.4 hold and suppose Φ is stochastically integrable with respect to W . Define the bounded operator $R : \mathcal{H}^* \rightarrow E^{**}$ by*

$$\langle x^*, Rh^* \rangle := \langle \Phi^* x^*, h^* \rangle_{\langle \mathcal{H}, \mathcal{H}^* \rangle}.$$

*Then $R = R_\Phi^{**}$, and both operators map \mathcal{H}^* into E .*

Proof. For all $f \in L^2(H)$ we have

$$\begin{aligned} \langle x^*, R_\Phi^{**} f \rangle &= \langle R_\Phi^* x^*, f \rangle_{\langle \mathcal{H}, \mathcal{H}^* \rangle} = \langle \Phi^* x^*, f \rangle_{\langle \mathcal{H}, \mathcal{H}^* \rangle} \\ (2.1) \quad &= \int_0^T [\Phi^*(t)x^*, f(t)]_H dt = \langle Rf, x^* \rangle. \end{aligned}$$

This proves that $R = R_{\Phi}^{**}$ as operators from H^* to E^{**} . To prove that R (and hence also R_{Φ}^{**}) maps \mathcal{H} into E it suffices to prove that for all $h \in \mathcal{H}$ we have

$$R_{\Phi}h = R_{\Phi}^{**}\phi_h$$

in E^{**} , and then to observe that R_{Φ} takes values in E . To prove this identity, note that for all $x^* \in E^*$ we have

$$\langle R_{\Phi}h, x^* \rangle_{\langle E, E^* \rangle} = [R_{\Phi}^*x^*, h]_{\mathcal{H}} = \langle R_{\Phi}^*x^*, \phi_h \rangle_{\langle \mathcal{H}, \mathcal{H}^* \rangle} = \langle x^*, R_{\Phi}^{**}\phi_h \rangle_{\langle E^*, E^{**} \rangle}.$$

□

It follows from the proposition that, under the stated assumptions, $t \mapsto \Phi(t)f(t)$ is Pettis integrable and, for all $f \in L^2(H)$,

$$(2.2) \quad Rf = \int_0^T \Phi(t)f(t) dt.$$

Now let A generate a C_0 -semigroup $(S(t))_{t \geq 0}$ on E and let $B \in \mathcal{L}(H, E)$ be a fixed operator. We are interested in the control problem

$$(2.3) \quad \begin{aligned} x' &= Ax + Bf, \\ x(0) &= x_0, \end{aligned}$$

where f is a ‘rough’ control, that is, we assume that $f \in \mathcal{H}^*$. For controls $f \in L^2(H)$ the mild solution x of the control problem at time T is given by

$$x(T) = S(T)x_0 + \int_0^T S(T-t)Bf(t) dt.$$

If we assume that $\Phi = S(T - \cdot)B$ satisfies the assumptions of Proposition 2.6, the identity (2.2) takes the form

$$Rf = \int_0^T S(T-t)Bf(t) dt.$$

Accordingly, for controls $f \in H^*$ we define the mild solution of the problem (2.3) at time T to be

$$x(T) = S(T)x_0 + Rf,$$

where $R : \mathcal{H}^* \rightarrow E$ is the map of Proposition 2.6.

Theorem 2.7. *Let Assumption 2.4 hold and let W be an \mathcal{H} -isonormal process. Suppose $S(T - \cdot)B$ is stochastically integrable with respect to W . Let \mathcal{G}_T be the RKHS associated with the stochastic integral $\int_0^T S(T-t)B dW(t)$. The following assertions are equivalent:*

- (i) $S(T)E \subseteq \mathcal{G}_T$;
- (ii) For all $x_0 \in E$ the problem

$$\begin{cases} x' &= Ax + Bf, \\ x(0) &= x_0, \end{cases}$$

is null controllable in time T with a control $f \in \mathcal{H}^*$.

- (iii) For all $x_0 \in E$ the laws of the processes solving the stochastic evolution equation

$$\begin{cases} dX(t) &= AX(t) + B dW, \\ X(0) &= x_0, \end{cases}$$

are mutually absolutely continuous.

Proof. (i) \Leftrightarrow (ii): The problem in (ii) is null controllable in time T with control $f \in \mathcal{H}^*$ if and only if $Rf = -S(T)x_0$. Now, $f = \phi_h$ for some $h \in \mathcal{H}$ and

$$Rf = R\phi_h = R_{\Phi}^{**}\phi_h = R_{\Phi}h.$$

Furthermore, the range of R_{Φ} equals the RKHS \mathcal{G}_T by Proposition 2.3. Combining things, we see that the problem in (ii) is null controllable in time T if and only if $S(T)x_0 \in \mathcal{G}_T$.

(i) \Leftrightarrow (iii): This is a special case of the Feldman-Hajek theorem on equivalent of Gaussian measures. \square

2.2. The smooth noise case. Next we consider the following dual assumption.

Assumption 2.8. The space $L^2(H)$ is continuously and densely included in \mathcal{H} .

We then have continuous and dense embeddings $\mathcal{H}^* \hookrightarrow L^2(H) \hookrightarrow \mathcal{H}$.

Definition 2.9. The function $\Phi : (0, T) \rightarrow \mathcal{L}(H, E)$ is said to be stochastically integrable with respect to the \mathcal{H} -isonormal process W if there exists an operator $R_{\Phi} \in \gamma(\mathcal{H}, E)$ such that $R_{\Phi}^*x^* = \Phi^*x^*$ for all $x^* \in E^*$. The random variable $W(R_{\Phi})$ is called the stochastic integral of Φ with respect to W , notation

$$\int_0^T \Phi dW = W(R_{\Phi}).$$

Note that, since we are assuming that the dual orbits $t \mapsto \Phi^*(t)x^*$ belong to $L^2(H)$, they automatically define elements of \mathcal{H} (unlike in the rough noise case).

Proposition 2.10. Let Assumption 2.8 hold and suppose Φ is stochastically integrable with respect to W . Define the bounded operator $R : \mathcal{H}^* \rightarrow E^{**}$ by

$$\langle x^*, Rh^* \rangle := \langle \Phi^*x^*, h^* \rangle_{\langle \mathcal{H}, \mathcal{H}^* \rangle}.$$

Then $R = R_{\Phi}^{**}$, and both operators map \mathcal{H}^* into E .

Proof. The proof follows the lines of that of Proposition 2.6 *verbatim*, except that the identities (2.1) hold only for elements $f \in \mathcal{H}^*$. \square

Now let A generate a C_0 -semigroup $(S(t))_{t \geq 0}$ on E and let $B \in \mathcal{L}(H, E)$ be a fixed operator. In the present setting, no ambiguities with regard to the definition of a mild solution for the control problem arise and we have the following result.

Theorem 2.11. Let Assumption 2.4 hold and let W be an \mathcal{H} -isonormal process. Suppose $S(T - \cdot)B$ is stochastically integrable with respect to W . Let \mathcal{G}_T be the RKHS associated with the stochastic integral $\int_0^T S(T - t)B dW(t)$. The following assertions are equivalent:

- (i) $S(T)E \subseteq \mathcal{G}_T$;

(ii) For all $x_0 \in E$ the problem

$$\begin{aligned} x' &= Ax + Bf, \\ x(0) &= x_0, \end{aligned}$$

is null controllable in time T with a control $f \in \mathcal{H}^*$.

(iii) For all $x_0 \in E$ the laws of the processes solving the stochastic evolution equation

$$\begin{aligned} dX(t) &= AX(t) + B dW, \\ X(0) &= x_0, \end{aligned}$$

are mutually equivalent.

Proof. The proof follows the lines of that of Theorem 2.7 *verbatim*. \square

3. FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES

3.1. Fractional Brownian motion. In the present section the results of the preceding part are applied to the fractional Ornstein-Uhlenbeck process, i.e. to the linear stochastic evolution equation in which the driving process is a classical space-dependent fractional Brownian motion (fBm) in time and white or, possibly, correlated in space. At first we recall standard definitions of these objects and explain how they may be understood in the framework developed above.

We begin with the case of a scalar-valued fBm $B^\beta = (B^\beta(t))_{t \in [0, T]}$ with Hurst parameter $\beta \in (0, 1)$. Following the approach taken in [3] we identify B^β with an \mathcal{H}_β -isonormal process W^β , the inner product of the real Hilbert space \mathcal{H}_β being given, for step functions $\varphi_1, \varphi_2 : [0, T] \rightarrow \mathbb{R}$, by

$$[\varphi_1, \varphi_2]_{\mathcal{H}_\beta} = \mathbb{E} W^\beta(\varphi_1) W^\beta(\varphi_2) = \mathbb{E} \int_0^T \varphi_1 dB^\beta \int_0^T \varphi_2 dB^\beta = [\mathcal{K}_\beta^* \varphi_1, \mathcal{K}_\beta^* \varphi_2]_{L^2(H)}.$$

Here, the operator $\mathcal{K}_\beta^* : \mathcal{H}_\beta \rightarrow L^2(H)$ is defined, for step functions $\varphi : [0, T] \rightarrow \mathbb{R}$, by

$$(3.1) \quad \mathcal{K}_\beta^* \varphi(t) = \varphi(t) K_\beta(T, t) + \int_t^T (\varphi(s) - \varphi(t)) \frac{\partial K_\beta}{\partial s}(s, t) ds,$$

where K_β is the real-valued kernel

$$(3.2) \quad K_\beta(t, s) = \frac{\tilde{c}_\beta (t-s)^{\beta-\frac{1}{2}}}{\Gamma(\beta+\frac{1}{2})} + \frac{\tilde{c}_\beta (\frac{1}{2}-\beta)}{\Gamma(\beta+\frac{1}{2})} \int_s^t (u-s)^{\beta-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-\beta}\right) du,$$

\tilde{c}_β being a constant depending only on β . We conclude that \mathcal{H}_β is the completion of the linear space of step functions with respect to the norm

$$(3.3) \quad \|\varphi\|_{\mathcal{H}_\beta} := \|\mathcal{K}_\beta^* \varphi\|_{L^2(H)}.$$

To give a more specific description of this space it is convenient to distinguish two cases, corresponding to the “rough” and “smooth” noise cases considered above.

For $0 < \beta < \frac{1}{2}$ (the rough noise case) we have

$$(\mathcal{K}_\beta^* \varphi)(t) = c_\beta t^{\frac{1}{2}-\beta} D_{T-}^{\frac{1}{2}-\beta} (u_{\beta-\frac{1}{2}} \varphi)(t), \quad \varphi \in \mathcal{H}_\beta,$$

where $u_\alpha(s) = s^\alpha$ and D_{T-}^α is the right-sided fractional derivative

$$(D_{T-}^\alpha \varphi)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{\varphi(t)}{(T-t)^\alpha} + \alpha \int_t^T \frac{\varphi(s) - \varphi(t)}{(s-t)^{\alpha+1}} ds \right)$$

and c_β is a constant depending only on β . It is not difficult to see that the space \mathcal{H}_β , as a set, may be identified as

$$\mathcal{H}_\beta = I_{T-}^{\frac{1}{2}-\beta}(L^2(0, T))$$

(cf. [1, Proposition 6]).

For $\frac{1}{2} < \beta < 1$ (the smooth noise case) we have

$$(\mathcal{H}_\beta^* \varphi)(t) = c_\beta t^{\frac{1}{2}-\beta} I_{T-}^{\beta-\frac{1}{2}}(u_{\beta-\frac{1}{2}} \varphi)(t), \quad \varphi \in \mathcal{H}_\beta,$$

where I_{T-}^α is the right-sided Riemann-Liouville integral,

$$(I_{T-}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \varphi(s) ds.$$

In this case we have a continuous dense embedding $L^2(H) \hookrightarrow \mathcal{H}_\beta$ and the operator \mathcal{H}_β^* restricts to a bounded operator on $L^2(H)$ (cf. [22]). Moreover, we have continuous dense embeddings

$$L^{\frac{1}{\beta}}(0, T, H) \hookrightarrow \overline{\Theta}_\beta \hookrightarrow \mathcal{H}_\beta,$$

where $\overline{\Theta}_\beta$ consists of those elements ψ from $L^1(0, T, H)$ such that

$$\|\psi\|_{\overline{\Theta}_\beta}^2 = \int_0^T \int_0^T |\psi(s)| \cdot |\psi(r)| \phi_\beta(r-s) dr ds < \infty$$

with $\phi_\beta(r) = (2\beta - 1)\beta|r|^{2\beta-2}$.

3.2. Cylindrical fractional Brownian motion. Let H be a real Hilbert space. An H -cylindrical fBm with Hurst parameter β is defined as a an $\mathcal{H}_\beta \overline{\otimes} H$ -isonormal process. Here, $\mathcal{H}_\beta \overline{\otimes} H$ denotes the Hilbert space tensor product of \mathcal{H}_β and H . Under the identification made at the beginning of the previous subsection, an \mathbb{R} -cylindrical fBm is just a classical scalar-valued fBm (with the same Hurst parameter).

If H has an orthonormal basis $(h_n)_{n \geq 1}$, one may think of an H -cylindrical fBm B^β as a formal series

$$W_t^\beta = \sum_{n=1}^{\infty} (B_n^\beta)_t h_n, \quad t \in \mathbb{R}_+,$$

where $(B_n^\beta)_{n \geq 1}$ is a sequence of independent fBm's with Hurst parameter β .

3.3. Ornstein-Uhlenbeck processes associated with a cylindrical fBm. Let H be a real Hilbert space, E a real Banach space, and let W^β an H -cylindrical fBm. We consider the equation

$$(3.4) \quad \begin{aligned} dZ_t^x &= AZ_t^x dt + B dW_t^\beta, \\ Z_0^x &= x, \end{aligned}$$

where A generates a strongly continuous semigroup $S = (S(t))_{t \geq 0}$ on E , the operator B is bounded and linear from H into E , and the initial datum x belongs to E . The solution is understood in the mild sense, i.e.

$$(3.5) \quad Z_t^x = S(t)x + \int_0^t S(t-r)B dW_r^\beta =: S(t)x + Z_t, \quad t \in [0, T],$$

provided the stochastic integral is well-defined. For Liouville cylindrical fBm, a necessary and sufficient condition for this is given in [3]; for $0 < \beta < \frac{1}{2}$ the same condition works for (classical) cylindrical fBm.

Let $\gamma(H, E)$ denote the space of γ -radonifying operators from H into E . For later reference we recall that if E is a Hilbert space, then $\gamma(H, E) = \mathcal{L}_2(H, E)$, the space of Hilbert-Schmidt operators from H to E , with equals norms.

A standard sufficient condition for existence and regularity of the so-called Ornstein-Uhlenbeck process $(Z_t)_{t \in [0, T]}$ is recalled in the following proposition. We shall need it only in the case that E is a Hilbert space and refer to [7, 8] for a proof for the case $\theta = 0$. For reasons of completeness we shall include a proof, which is an adaptation of the argument in [3, Theorem 5.5] (where more details are provided).

If the semigroup S is analytic we may find $z_0 > 0$ sufficiently large such that the fractional powers of $z_0 - A$ exist (z_0 is fixed in the sequel) and we denote by E^θ the domain of the fractional power $(z_0 - A)^\theta$ equipped with the graph norm.

Proposition 3.1. *Suppose S is a strongly continuous analytic semigroup on the real Hilbert space E , let $B \in \mathcal{L}(H, E)$ be bounded, and assume that for some $\theta \in [0, 1)$ and $\lambda \geq 0$ we have*

$$(3.6) \quad \|S(t)B\|_{\mathcal{L}_2(H, E^\theta)} \leq ct^{-\lambda}, \quad t \in (0, T],$$

for some $c \geq 0$. If

$$\delta + \theta + \lambda < \beta,$$

the stochastic convolution process $(Z_t)_{t \in [0, T]}$ defined by (3.5) is well-defined and has a modification with paths in $C^\delta([0, T]; E^\theta)$.

Proof. Fix $0 \leq s \leq t \leq T$. By the triangle inequality in $L^2(\Omega; E)$,

$$(3.7) \quad (\mathbb{E} \|Z(t) - Z(s)\|_{E^\theta}^2)^{\frac{1}{2}} \leq \left(\mathbb{E} \left\| \int_0^s [S(t-r) - S(s-r)]B dW^\beta(r) \right\|_{E^\theta}^2 \right)^{\frac{1}{2}} \\ + \left(\mathbb{E} \left\| \int_s^t S(t-r)B dW^\beta(r) \right\|_{E^\theta}^2 \right)^{\frac{1}{2}}.$$

We shall estimate both terms separately.

Fix $\tau \geq 0$ such that

$$\lambda < \tau < \beta - \delta - \theta.$$

Then (with generic constants c)

$$\|(z_0 - A)^{-\tau} B\|_{\mathcal{L}_2(H, E^\theta)} \leq c \int_0^\infty t^{\tau-1} \|S(t)B\|_{\mathcal{L}_2(H, E^\theta)} dt \leq c \int_0^\infty t^{\tau-1-\lambda} dr < \infty.$$

For the first term in (3.7) we have, for any $\varepsilon > 0$ such that $\delta + \tau + \theta + \varepsilon < \beta$,

$$\begin{aligned} & \mathbb{E} \left\| \int_0^s [S(t-r) - S(s-r)]B dW^\beta(r) \right\|_{E^\theta}^2 \\ & \simeq \mathbb{E} \left\| \int_0^s (s-r)^{\delta+\tau+\theta+\varepsilon} (z_0 - A)^{\delta+\tau+\theta} S(s-r) \right. \\ & \quad \times (s-r)^{-\delta-\tau-\theta-\varepsilon} [S(t-s) - I](z_0 - A)^{-\delta-\tau} B dW^\beta(r) \left. \right\|^2 \\ & \leq c^2 \mathbb{E} \left\| \int_0^s (s-r)^{-\delta-\tau-\theta-\varepsilon} [S(t-s) - I](z_0 - A)^{-\delta-\tau} B dW^\beta(r) \right\|^2 \\ & = c^2 \left\| [S(t-s) - I](z_0 - A)^{-\delta-\tau} B \right\|_{\mathcal{L}_2(H, E)}^2 \mathbb{E} \left| \int_0^s (s-r)^{-\delta-\tau-\theta-\varepsilon} dW^\beta(r) \right|^2 \end{aligned}$$

$$\begin{aligned}
&= c^2 s^{2\beta-2\delta-2\tau-2\theta-2\varepsilon} \| [S(t-s) - I](z_0 - A)^{-\delta} (z_0 - A)^{-\tau} B \|_{\mathcal{L}_2(H,E)}^2 \\
&\leq c^2 (t-s)^{2\delta} \| (z_0 - A)^{-\tau} B \|_{\mathcal{L}_2(H,E)}^2,
\end{aligned}$$

where the numerical value of c changes from line to line (and is allowed to depend on T).

Similarly,

$$\begin{aligned}
&\mathbb{E} \left\| \int_s^t S(t-r) B dW^\beta(r) \right\|_{E^\theta}^2 \\
&\approx \mathbb{E} \left\| \int_s^t (t-r)^{\theta+\tau+\varepsilon} (z_0 - A)^{\theta+\tau} S(t-r) (t-r)^{-\theta-\tau-\varepsilon} (z_0 - A)^{-\tau} B dW^\beta(r) \right\|^2 \\
&\leq c^2 \mathbb{E} \left\| \int_s^t (t-r)^{-\tau-\theta-\varepsilon} (z_0 - A)^{-\tau} B dW^\beta(r) \right\|^2 \\
&= c^2 \| (z_0 - A)^{-\tau} B \|_{\mathcal{L}_2(H,E)}^2 \mathbb{E} \left| \int_s^t (t-r)^{-\tau-\theta-\varepsilon} dW^\beta(r) \right|^2 \\
&\leq c^2 \| (z_0 - A)^{-\tau} B \|_{\mathcal{L}_2(H,E)}^2 (t-s)^{2\beta-2\tau-2\theta-2\varepsilon} \\
&\leq c^2 \| (z_0 - A)^{-\tau} B \|_{\mathcal{L}_2(H,E)}^2 (t-s)^{2\delta}.
\end{aligned}$$

Combining these estimates, this gives

$$\mathbb{E} \| Z(t) - Z(s) \|_{E^\theta}^2 \leq c^2 \| (z_0 - A)^{-\tau} B \|_{\mathcal{L}_2(H,E)}^2 (t-s)^{2\delta}.$$

The assertion now follows from a routine application of the Kahane-Khintchine inequalities (to pass from moments of order 2 to moments of order p) and the Kolmogorov-Chentsov continuity theorem. \square

Now we are ready to formulate the main result for the fractional Ornstein-Uhlenbeck process.

Theorem 3.2. *Suppose A generates a strongly continuous analytic semigroup S on E . Suppose furthermore that $B \in \mathcal{L}(H, E)$ is bounded and injective. Assume:*

- (i) $\|S(t)B\|_{\gamma(H,E)} \leq ct^{-\lambda}$, $t \in (0, 1)$, for some $\lambda \in [0, \beta)$,
- (ii) $\text{Range}(B) \supset \text{Dom}((z_0 - A)^\mu)$ for some $\mu \in [0, 1 - \beta)$.

Then there exists a continuous mild solution Z^x to the equation (3.4), and for each $T > 0$ the probability laws of Z_T^x , $x \in E$, are equivalent.

Proof. Since B is injective and $S(t)$ maps E into $D(A)$ for all $t > 0$, the operators $B^{-1}S(t)$ are well-defined, and by the closed graph theorem they are bounded. By (ii),

$$(3.8) \quad \|B^{-1}S(t)\|_{\mathcal{L}(E,H)} \leq \|B^{-1}(z - A)^{-\mu}\|_{\mathcal{L}(E,H)} \|(z - A)^\mu S(t)\|_{\mathcal{L}(E)} \leq \frac{c}{t^\mu},$$

where the last step uses the analyticity of the semigroup S .

a) *The case $0 < \beta < \frac{1}{2}$.* By Proposition 3.1 and Theorem 2.7 it remains to show the null controllability of the equation

$$\dot{y} = Ay + Bu$$

on $[0, T]$ for the space of controls \mathcal{H}^* . Note that

$$\|\varphi\|_{\mathcal{H}} = \|\mathcal{K}_\beta^* \varphi\|_{L^2(H)}.$$

Hence for $\varphi \in \mathcal{H}^*$ we have

$$\begin{aligned} \|\varphi\|_{\mathcal{H}^*} &= \sup_{\|h\|_{\mathcal{H}} \leq 1} |[\varphi, h]_{L^2(H)}| = \sup_{\|g\|_{L^2(H)} \leq 1} \left| \int_0^T [\varphi, (\mathcal{K}_\beta^*)^{-1}g]_H ds \right| \\ &= \sup_{\|g\|_{L^2(H)} \leq 1} \left| \int_0^T [\mathcal{K}_\beta^{-1}\varphi, g]_H ds \right| = \left(\int_0^T \|\mathcal{K}_\beta^{-1}\varphi\|_H^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

taking into account that $((\mathcal{K}_\beta^*)^{-1})^* = \mathcal{K}_\beta^{-1} = c_\beta u_{\beta-\frac{1}{2}} I_{0+}^{\frac{1}{2}-\beta} u_{\frac{1}{2}-\beta}$, where c_β is a constant and I_{0+}^α is the left-sided fractional integral,

$$(I_{0+}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds, \quad \varphi \in L^2(H).$$

Moreover, by (3.8) and the fact that the assumption on μ implies $\beta - \frac{1}{2} + \mu < 1$,

$$\begin{aligned} \|\mathcal{K}_\beta^{-1} B^{-1} S(\cdot)(t)\| &\leq ct^{\beta-\frac{1}{2}} \int_0^t \frac{\|B^{-1} S(s)\| s^{\frac{1}{2}-\beta}}{(t-s)^{\beta+\frac{1}{2}}} ds \\ &\leq ct^{\beta-\frac{1}{2}} \int_0^t \frac{s^{\frac{1}{2}-\beta-\mu}}{(t-s)^{\beta+\frac{1}{2}}} ds = \frac{c}{t^{\beta-\frac{1}{2}+\mu}}, \end{aligned}$$

where $c \geq 0$ is a generic constant whose value is allowed to change from line to line. Since by assumption $\mu < 1 - \beta$, this shows that $\mathcal{K}_\beta^{-1}(B^{-1} S(T - \cdot)) \in L^2(H)$, and therefore the control $\hat{u}_s(t) = -\frac{1}{T} B^{-1} S(t)x$ steering x to zero belongs to the space \mathcal{H}^* .

b) *The case $\frac{1}{2} < \beta < 1$.* As in the previous case we only need to show the null controllability of the equation

$$\dot{y} = A + Bu$$

on $[0, T]$ for the space of controls \mathcal{H}^* . Using Theorem 2.11 and proceeding as in a) it suffices to show that

$$\int_0^T \|\mathcal{K}_\beta^{-1} \varphi(s)\|_H^2 ds < \infty$$

where $\varphi(s) := B^{-1} S(s)x$, $x \in E$, is fixed and

$$\mathcal{K}_\beta^{-1} = c_\beta u_{\beta-\frac{1}{2}} D_{0+}^{\beta-\frac{1}{2}} u_{\frac{1}{2}-\beta},$$

and D_{0+}^α denotes the left-sided fractional derivative.

$$(D_{0+}^\alpha \psi)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{\psi(t)}{t^\alpha} + \alpha \int_0^t \frac{\psi(t) - \psi(s)}{(t-s)^{\alpha+1}} ds \right), \quad \psi \in \mathcal{H}_\beta.$$

We have, for $\beta - \frac{1}{2} < \delta < \frac{1}{2}$ and $z_0 > 0$ large enough,

$$\begin{aligned} &\int_0^T \|\mathcal{K}_\beta^{-1} \varphi(t)\|_H^2 dt \\ &\leq c \int_0^T \frac{\|B^{-1} S(t)x\|_H^2 t^{2\beta-1} \cdot t^{1-2\beta}}{t^{2\beta-1}} dt \\ &\quad + c \int_0^T t^{2\beta-1} \left(\int_0^t \frac{\|B^{-1} S(t)x - B^{-1} S(s)x\|_H s^{\frac{1}{2}-\beta}}{(t-s)^{\beta+\frac{1}{2}}} ds \right)^2 dt \end{aligned}$$

$$\begin{aligned}
&\leq c \int_0^T \left[\frac{1}{t^{2\beta-1+2\mu}} + t^{2\beta-1} \times \right. \\
&\quad \left. \left(\int_0^t \frac{\|B^{-1}S(\frac{s}{4})\| \|(z_0 - A)^{-\delta}(S(t - \frac{3}{4}s) - S(\frac{s}{4}))\|_H \|(z_0 - A)^{\delta}S(\frac{s}{2})x\|}{(t-s)^{\beta+\frac{1}{2}}s^{\beta-\frac{1}{2}}} ds \right)^2 \right] dt \\
&\leq c \int_0^T \left[\frac{1}{t^{2\beta-1+2\mu}} + t^{2\beta-1} \left(\int_0^t \frac{1}{(t-s)^{\beta+\frac{1}{2}-\delta}s^{\delta+\mu+\beta-\frac{1}{2}}} dr \right)^2 \right] dt
\end{aligned}$$

for generic constants c , where we used (3.8) and analyticity of the semigroup S . It follows that

$$\begin{aligned}
\int_0^T \|\mathcal{K}_{\beta}^{-1}\varphi(t)\|_H^2 dt &\leq c \int_0^T \left[\frac{1}{t^{2\beta-1+2\mu}} + t^{2\beta-1} \left(\frac{1}{t^{2\beta+\mu-1}} \right)^2 \right] dt \\
&= c \int_0^T \frac{1}{t^{2\beta-1+2\mu}} dt,
\end{aligned}$$

which is finite since $\mu < 1 - \beta$. Therefore, the control $\hat{u}(t) := -\frac{1}{T}B^{-1}S(t)x$ steering the solution from x to zero belongs to \mathcal{H}^* as required. \square

Example 3.3 ($2m$ -th order parabolic equation). We consider the problem

$$\begin{cases} \frac{\partial y}{\partial t}(t, \xi) = (L_{2m}y)(t, \xi) + \eta^{\beta}(t, \xi), & (t, \xi) \in (0, T) \times D, \\ y(0, \xi) = x(\xi), & \xi \in D, \end{cases}$$

with the Dirichlet boundary conditions

$$\frac{\partial^k y}{\partial \nu^k}(t, \xi) = 0, \quad k = 0, \dots, m-1, \quad (t, \xi) \in (0, T) \times \partial D$$

where $\frac{\partial}{\partial \nu}$ denotes the co-normal derivative, $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary and

$$L_{2m} = \sum_{|\alpha| \leq 2m} a_{\alpha}(\xi) D^{\alpha}$$

with $a_{\alpha} \in C_b^{\infty}(D)$ is a uniformly elliptic operator on D . We let A denote its realisation in $E = L^2(D)$. The Gaussian noise is fractional in time and is modelled as

$$\eta^{\beta}(t, \xi) = \left(B \frac{\partial}{\partial t} W^{\beta}(t, \cdot) \right)(\xi),$$

where B is a given bounded injective operator on $H = E = L^2(D)$.

Suppose that condition (ii) of the theorem holds with exponent $\mu \in [0, 1 - \beta)$ (where we may put $z_0 = 0$), suppose that there exists $\mu' \in [0, \mu]$ such that

$$\text{Range}(B) \subset \text{Dom}(-A)^{\mu'}$$

and assume that

$$(3.9) \quad \frac{d}{4m} < \beta + \mu'.$$

Then condition (i) of the theorem is satisfied as well. Indeed, as is well known, $(-A)^{-\rho}$ is Hilbert-Schmidt for any $\rho > \frac{d}{4m}$, so

$$\begin{aligned}
\|S(t)B\|_{\mathcal{L}_2(H, E)} &\leq c \|S(t)(-A)^{-\mu'}\|_{\mathcal{L}_2(E)} \\
&\leq c \|(-A)^{-\rho}\|_{\mathcal{L}_2(E)} \cdot \|(-A)^{\rho-\mu'} S(t)\|_{\mathcal{L}(E)} \leq c \frac{c}{t^{\rho-\mu'}}.
\end{aligned}$$

It follows that (i) is satisfied. Thus, equivalence of the laws of Z_T^x , $x \in E$, is obtained if (3.9) and the condition (ii) of the theorem hold. Note that (3.9) is always satisfied if $\frac{d}{4m} < \beta$.

For example, for the stochastic heat equation ($m = 1$) with the noise fractional in time and white in space ($\mu = \mu' = 0$) the result holds if $\beta > \frac{d}{4}$.

The condition (ii) of Theorem 3.2 is just sufficient and not necessary, as can be seen in the following example.

Example 3.4. In this example we take $H = E$ to be a separable real Hilbert space with orthonormal basis $(e_n)_{n \geq 1}$ and define the operators A and B by

$$Ae_n = -\alpha_n e_n, \quad Be_n = \sqrt{\lambda_n} e_n, \quad n \geq 1,$$

with $0 < \lambda_n \leq \lambda_0$ and $0 < \alpha_1 \leq \alpha_2 \leq \dots \rightarrow \infty$. Let H_T denote the reproducing kernel Hilbert space associated with the covariance operator Q_T of the Gaussian random variable $Z_T = \int_0^T S(T-t)B dW_t^\beta$. By the results of Section 2, equivalence of laws for Z_T^x , $x \in E$, holds if and only if the range of $S(T)$ is contained in H_T . Since $S(T)$ is self-adjoint, this happens if and only if there is a constant $c \geq 0$ such that

$$\|S(T)x\|^2 \leq c \langle Q_T x, x \rangle, \quad x \in E.$$

Under the above assumptions, this is equivalent to

$$(3.10) \quad e^{-2\alpha_n T} \leq c q_n, \quad n \geq 1,$$

where $Q_T e_n = q_n e_n$. It is easily seen that (3.10) holds if and only if

$$(3.11) \quad \frac{\alpha_n^{2\beta}}{\lambda_n} e^{-2\alpha_n T}$$

is bounded. This follows from the fact that there exist some constants $C_1, C_2 \geq 0$ such that

$$c_1 \frac{\lambda_n}{\alpha_n^{2\beta}} \leq q_n \leq c_2 \frac{\lambda_n}{\alpha_n^{2\beta}}$$

For $\beta > \frac{1}{2}$, this was proved in [7]. If $0 < \beta < \frac{1}{2}$ we have

$$q_n(T) = \langle Q_T e_n, e_n \rangle = \lambda_n \int_0^T \|\mathcal{K}_\beta^* \psi_n(t)\|_H^2 dt$$

where $\psi_n(t) = e^{-\alpha_n t}$. Furthermore,

$$\begin{aligned} \int_0^T \|\mathcal{K}_\beta^* \psi_n(t)\|_H^2 dt &\cong \|\psi_n\|_{H^{\frac{1}{2}-\beta}(0,T)}^2 = \int_0^T \int_0^T \frac{|e^{-\alpha_n r} - e^{-\alpha_n s}|^2}{|r-s|^{2-2\beta}} dr ds \\ &= \frac{1}{\alpha_n^{2\beta}} \int_0^{\alpha_n T} \int_0^{\alpha_n T} \xi(t,s) dt ds \end{aligned}$$

where $\xi(t,s) = \frac{|e^{-t} - e^{-s}|^2}{|t-s|^{2-2\beta}}$, and the conclusion easily follows.

Clearly, (3.11) is implied by, but does not necessarily imply, condition (ii) of the theorem.

Example 3.5. Consider the 1D stochastic parabolic equation with inhomogeneous Neumann boundary conditions of fractional noise type, formally written as

$$(3.12) \quad \begin{cases} \frac{\partial y}{\partial t}(t, \xi) = \frac{\partial}{\partial \xi} \left(p(\xi) \frac{\partial y}{\partial \xi} \right)(t, \xi) + q(\xi) y(t, \xi), & (t, \xi) \in (0, T) \times (0, 1), \\ y(0, \xi) = x(\xi), & \xi \in (0, 1), \\ \frac{\partial y}{\partial \xi}(t, 0) = \sigma_0 \dot{B}_1^\beta(t), \quad \frac{\partial y}{\partial \xi}(t, 1) = \sigma_1 \dot{B}_2^\beta(t), & t \in (0, T), \end{cases}$$

where $W^\beta = (B_1^\beta, B_2^\beta)$ is a two-dimensional standard fBm with the Hurst parameter $\beta \in (0, 1)$, σ_0, σ_1 are real constants, the initial datum x belongs to $L^2(0, 1)$, $p \in C^2[0, 1]$ is strictly positive, and $q \in C[0, 1]$.

It is standard to rewrite the formal equation (3.12) in the infinite-dimensional form considered in the previous section (see e.g. [6, 7] and references therein),

$$(3.13) \quad \begin{cases} dX_t = AX_t dt + B dW_t^\beta, & t \in [0, T], \\ X_0 = x. \end{cases}$$

Here A is the realisation in the space $E = L^2(0, 1)$ of the partial differential operator

$$A = \frac{\partial}{\partial \xi} \left(p(\cdot) \frac{\partial}{\partial \xi} \right) + qI$$

with domain

$$\text{Dom}(A) = \{\varphi \in E; \varphi, \varphi' \text{ absolutely continuous, } \varphi'' \in E, \varphi'(0) = \varphi'(1) = 0\}.$$

The operator A is uniformly elliptic with homogeneous Neumann boundary conditions and it is well known that it generates a strongly continuous and analytic semigroup S on E . In order to define the operator B we set $H = \mathbb{R}^2$. Fix a constant $k > \max q$ and consider the second order boundary value problem

$$\begin{aligned} kz - Az &= 0 \quad \text{on } (0, 1) \\ \frac{\partial z}{\partial \xi}(j) &= u_j, \quad j \in \{0, 1\}, \end{aligned}$$

for $(u_0, u_1) \in \mathbb{R}^2$. This problem has a unique solution for every $(u_0, u_1) \in \mathbb{R}^2$. The Neumann map $N : (u_0, u_1) \mapsto z$ is an element of the space $\mathcal{L}(\mathbb{R}^2, E^\varepsilon)$ for arbitrary $\varepsilon < \frac{3}{4}$ (see [6, 13] for details). Setting $B = \hat{A}N$, $\hat{A} \in \mathcal{L}(E^\varepsilon, E^{\varepsilon-1})$ is the isomorphic extension of the operator $kI - A$, we thus have $B \in \mathcal{L}(\mathbb{R}^2, E^{\varepsilon-1}) = \mathcal{L}(H, E^{\varepsilon-1})$. Now, the infinite-dimensional form (the mild solution) of the equation (3.13) reads

$$(3.14) \quad X_t^x = S(t)x + \int_0^t S(t-r)B dW^\beta(r), \quad t \in [0, T],$$

(here the extension of the semigroup $S(t)$, $t > 0$ to the space $\mathcal{L}(E^{\varepsilon-1}, E)$ is denoted again by $S(t)$) which fits in the framework developed in Section 2 with the spaces $H = \mathbb{R}^2$ and $E_1 = E^{\varepsilon-1}$ (for more detailed justification of this approach we refer to [6, 7, 13]).

Assume that $\beta \in (\frac{1}{4}, 1)$ and fix $\varepsilon \in (1 - \beta, \frac{3}{4})$. Then by the analyticity of the semigroup S ,

$$\|S(t)B\|_{\mathcal{L}_2(H, E)} \leq c_1 \|S(t)\|_{\mathcal{L}(E^{\varepsilon-1}, E)} \|B\|_{\mathcal{L}(\mathbb{R}^2, E^{\varepsilon-1})} \leq c_2 t^{\varepsilon-1}, \quad t \in (0, T],$$

for some constants c_1, c_2 , so by [7] (or Proposition 3.1) the mild solution (3.14) is a well-defined E -continuous process. Our aim is to investigate the equivalence of

probability distributions μ_T^x of solutions X_T^x corresponding to initial datum $x \in E$. To this end, we use the null controllability result of Fattorini and Russel [10] for the controlled equation

$$(3.15) \quad \begin{cases} \frac{\partial y}{\partial t}(t, \xi) = (Ay)(t, \xi), & t \in (0, T), \xi \in (0, 1), \\ y(0, \xi) = x(\xi), & \xi \in (0, 1), \\ \frac{\partial}{\partial \xi} y(t, 0) = u_0(t), \quad \frac{\partial}{\partial \xi} y(t, 1) = u_1(t), & t \in (0, T), \end{cases}$$

with $u = (u_1, u_2) \in L^2(0, T)$. First, note that the operator A is self-adjoint and possesses a sequence $(-\lambda_n)_{n \geq 1}$ of real eigenvalues such that

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

and $\lim_{n \rightarrow \infty} \lambda_n = +\infty$. Moreover, there is a real constant α such that

$$(3.16) \quad \lambda_n = \frac{\pi^2}{L^2}(n + \alpha)^2 + O(1), \quad n \rightarrow \infty,$$

where $L = \int_0^1 p^{-\frac{1}{2}}(z) dz$ (cf. [4, 26]). Denoting by $(e_n)_{n \geq 1}$ the normalized eigenfunctions corresponding to $(\lambda_n)_{n \geq 1}$, the solution to the equation (3.15) may be expressed by the expansion $y(T) = \sum_{n=1}^{\infty} y_n(T) e_n$ in $L^2(0, 1)$, where

$$y_n(T) = e^{-\lambda_n T} x_n + \int_0^T e^{-\lambda_n(T-t)} \beta_0^n u_0(t) dt + \int_0^T e^{-\lambda_n(T-t)} \beta_1^n u_1(t) dt$$

with $x_n = \langle x, e_n \rangle$ and

$$\beta_0^n = -p(0)\sigma_0 e_n(0), \quad \beta_1^n = -p(1)\sigma_1 e_n(1)$$

(cf. [10]). It is clear that (3.15) is not controllable in any usual sense if $\sigma_0 = \sigma_1 = 0$. Assume that at least one of the constants σ_0, σ_1 is not zero (for instance, $\sigma_0 \neq 0$). Set

$$c_n = \frac{e^{-\lambda_n T} x_n}{\beta_0^n}.$$

It is known that $\beta_0^n \sim \text{const} \sqrt{|\lambda_n|}$ (cf. [10]). Therefore, taking $\eta > 0$ arbitrary:

$$\sum_{n=0}^{\infty} |c_n \lambda_n| \exp\{(L + \eta) \sqrt{|\lambda_n|}\} \leq c \sum_{n=0}^{\infty} |x_n| \sqrt{|\lambda_n|} e^{-\lambda_n T} \exp\{(L + \eta) \sqrt{|\lambda_n|}\} < \infty$$

by (3.16). Hence by [10, Corollary 3.2], there exists a solution $h \in L^2(0, T)$ to the moment problem

$$\int_0^T e^{-\lambda_n t} h(t) dt = \lambda_n c_n, \quad n \geq 1.$$

It follows that $u_0(t) := \int_0^t h(s) ds$ solves the moment problem

$$\int_0^T e^{-\lambda_n t} u_0(t) dt = c_n$$

and therefore, the control $u(t) = (u_0(t), 0)$ steers the point x to zero at $t = T$. Obviously, $u \in W^{1,2}(0, T, \mathbb{R}^2)$ and it is easy to verify that $W^{1,2}(0, T, \mathbb{R}^2) \subset \mathcal{H}^*$ for each value of the Hurst parameter $\beta \in (0, 1)$. Summarizing, by Theorems 2.7 and 2.11 we obtain that for each $\beta \in (0, 1)$ the measures μ_T^x , $x \in E$, are equivalent whenever $\sigma_0^2 + \sigma_1^2 \neq 0$.

4. STRONG FELLER PROPERTY FOR SEMILINEAR EQUATIONS

In this section we present some applications of the general results from the previous part to stochastic semilinear equations with additive fractional noise. It is shown that the null controllability of the deterministic equation

$$(4.1) \quad y' = Ay + Bu$$

in the appropriate sense is equivalent to the continuous dependence of probability laws of solutions to the corresponding stochastic semilinear equation on the initial datum. The latter property in the Markovian case is called the *strong Feller property*.

Consider the semilinear equation

$$(4.2) \quad \begin{cases} dX_t^x = (AX_t^x + F(X_t^x)) dt + B dW_t^\beta, & t \in (0, T), \\ X_0^x = x \end{cases}$$

in the space E , which is here for simplicity assumed to be a real separable Hilbert space (cf. Remark 4.6 below). The operators A and B and the driving noise W_t^β are the same as in the linear case (i.e. W_t^β is an H -isonormal Gaussian process representing the H -cylindrical fBm with the Hurst parameter $\beta \in (0, 1)$). The operator $A : \text{Dom}(A) \subset E \rightarrow E$ is assumed to generate an analytic semigroup $S = (S(t))_{t \geq 0}$ on E and the condition (3.6) of Proposition 3.1 is supposed to hold. Under these assumptions, for each initial datum $x \in E$ the mild solution $(Z_t^x)_{t \in [0, T]}$ of the linear equation (3.4) exists and has a modification with paths in $C^\delta([0, T], E)$ for all $0 \leq \delta < \beta - \lambda$. Let us now in addition assume that $B \in \mathcal{L}(H, E)$ is injective and that the range of the nonlinear function $F : E \rightarrow E$ is contained in the range of B . This allows us to define the function $G : E \rightarrow H$, $G := B^{-1}F$. We impose the following conditions on G :

(G) The function $G : E \rightarrow H$ is continuous and has at most linear growth, i.e.

$$(4.3) \quad \|G(x)\|_H \leq k(1 + \|x\|_E), \quad x \in E,$$

for some $k \geq 0$. If $\beta > \frac{1}{2}$ we make the additional Hölder continuity assumption

$$(4.4) \quad \|G(x) - G(y)\|_H \leq k\|x - y\|_E^\alpha, \quad x, y \in E,$$

for some $k \geq 0$ and

$$(4.5) \quad \alpha > \frac{\beta - \frac{1}{2}}{\beta - \lambda}.$$

Define the integral operator \mathbb{K}_β induced by the kernel $K_\beta(t, s)$ (cf. (3.2)),

$$(\mathbb{K}_\beta \varphi)(t) = \int_0^t K_\beta(t, s) \varphi(s) ds$$

for $\varphi \in L^2(0, T, H)$. By [23], the operator

$$K_\beta : L^2(0, T, H) \rightarrow I_{0+}^{\beta+\frac{1}{2}}(L^2(0, T, H))$$

is a bijection and its inverse \mathbb{K}_β^{-1} may be expressed, for $\varphi \in I_{0+}^{\beta+\frac{1}{2}}(L^2(0, T, H))$, as

$$(\mathbb{K}_\beta^{-1} \varphi)(t) = c_\beta t^{\frac{1}{2}-\beta} D_{0+}^{\frac{1}{2}-\beta} (t_{\beta-\frac{1}{2}} D_{0+}^{2\beta} \varphi)(t)$$

for $\beta \in (0, \frac{1}{2})$ and

$$(4.6) \quad (\mathbb{K}_\beta^{-1}\varphi)(t) = c_\beta t_{\beta-\frac{1}{2}} D_{0+}^{\beta-\frac{1}{2}}(t_{\frac{1}{2}-\beta} D\varphi)(t)$$

for $\beta \in (\frac{1}{2}, 1)$. If moreover $\varphi \in W^{1,2}(0, T, H)$ we have

$$(4.7) \quad (\mathbb{K}_\beta^{-1}\varphi)(t) = c_\beta t_{\beta-\frac{1}{2}} I_{0+}^{\frac{1}{2}-\beta}(t_{\frac{1}{2}-\beta} D\varphi)(t)$$

for $\beta \in (0, \frac{1}{2})$ (here c_β is a positive constant depending only on $\beta \in (0, 1)$) (cf. [20]).

By (3.3), the Gaussian process \widetilde{W} defined as $\widetilde{W}(h) := W^\beta((\mathcal{K}_\beta^*)^{-1}h)$, where \mathcal{K}_β^* is the operator defined in (3.1), is isonormal on $\mathcal{H} = L^2(0, T, H)$, i.e. it is the classical white noise (see also [2]). The following result has been proved in [9].

Proposition 4.1. *Assume that (3.6) and (G) are satisfied. Then the equation (4.2) has a weak (in the probabilistic sense) solution $(X(t))$ satisfying $X(0) = x$ which is weakly unique. Moreover, for each $x \in E$ and $T > 0$ the probability laws μ_T^x and ν_T^x are equivalent, where $\mu_T^x = \text{Law}(Z_T^x)$ and $\nu_T^x = \text{Law}(X_T^x)$, and the density is given by*

$$\widetilde{\mathbb{E}}\varphi(X_T^x) = \mathbb{E}\varphi(Z_T^x)\rho_T(x)$$

where $\widetilde{\mathbb{E}}$ is the expectation with respect to the probability space where the process (X_t^x) is defined, $\varphi : E \rightarrow \mathbb{R}$ is bounded Borel measurable and

$$\begin{aligned} \rho_T(x) := \exp \Big\{ & \int_0^T \left\langle \mathbb{K}_\beta^{-1} \left(\int_0^\cdot G(Z_s^x) ds \right) (t) d\widetilde{W} \right\rangle_H \\ & - \frac{1}{2} \int_0^T \left\| \mathbb{K}_\beta^{-1} \left(\int_0^\cdot G(Z_s^x) ds \right) (t) \right\|_H^2 dt \Big\}. \end{aligned}$$

Let $\mathcal{B}(E)$ denote the σ -algebra of Borel sets in E and τ the topology of pointwise convergence in the space of finite signed measures on $\mathcal{B}(E)$. Thus a net $(\mu_\gamma)_{\gamma \in \Gamma}$ converges to μ in τ if and only if $\lim_{\gamma \in \Gamma} \mu_\gamma(C) = \mu(C)$ for each $C \in \mathcal{B}(E)$. Now we formulate the main result of this section.

Theorem 4.2. *Assume (3.6) and (G). Then for each $T > 0$ the following statements are equivalent:*

- (i) *The controlled deterministic system (4.1) is null controllable in time T .*
- (ii) *The measures μ_T^x , $x \in E$, are equivalent.*
- (iii) *The measures ν_T^x , $x \in E$, are equivalent.*
- (iv) *$\nu_T^{x_n} \rightarrow \nu_T^x$ in τ whenever $x_n \rightarrow x$ in E (strong Feller property).*

In the proof of Theorem 4.2 we use the following lemma:

Lemma 4.3. *Assume (3.6) and (G). Then for each $T > 0$*

$$\rho_T(x_n) \rightarrow \rho_T(x) \text{ in } \mathbb{P} \text{ if } x_n \rightarrow x \text{ in } E.$$

Proof. From the proofs of [9, Theorems 3.3 and 3.4] it easily follows that

$$\mathbb{E} \int_0^T \left\| \mathbb{K}_\beta^{-1} \left(\int_0^\cdot G(Z_s^x) ds \right) (t) \right\|_H^2 dt < \infty, \quad x \in E.$$

Hence it suffices to show that

$$(4.8) \quad \mathbb{E} \int_0^T \left\| \mathbb{K}_\beta^{-1} \left(\int_0^\cdot (G(Z_s^{x_n}) - G(Z_s^x)) ds \right) (t) \right\|_H^2 dt \rightarrow 0$$

whenever we have $x_n \rightarrow x$ in E . We will show (4.8) separately for the cases $\beta \in (0, \frac{1}{2})$ and $\beta \in (\frac{1}{2}, 1)$.

First, consider the case $\beta \in (0, \frac{1}{2})$. By (4.7) we have

$$\begin{aligned} & \mathbb{E} \left\| \mathbb{K}_\beta^{-1} \left(\int_0^\cdot (G(Z_s^{x_n}) - G(Z_s^x)) ds \right) \right\|_{L^2(0,T,H)}^2 \\ & \leq c_\beta^2 \mathbb{E} \int_0^T \left(s^{\beta-\frac{1}{2}} \left\| \int_0^s r^{\frac{1}{2}-\beta} (s-r)^{-\frac{1}{2}-\beta} (G(Z_r^{x_n}) - G(Z_r^x)) dr \right\|_H \right)^2 ds. \end{aligned}$$

By continuity of G we have $G(Z_r^{x_n}) - G(Z_r^x) \rightarrow 0$ for each $r \in [0, T]$ \mathbb{P} -almost surely, and (4.3) yields

$$(4.9) \quad \|G(Z_r^{x_n}) - G(Z_r^x)\|_H \leq 2k(1 + \|S(r)x_n\|_E + \|Z_r^0\|) \leq L(1 + \|\tilde{Z}\|_{C([0,T],E)})$$

where $L \geq 0$ is a constant independent of $n \in \mathbb{N}$ and $r \in [0, T]$. Hence we obtain (4.8) by the dominated convergence theorem.

Now, consider the case $\beta \in (\frac{1}{2}, 1)$. By (4.6) it follows that

$$\begin{aligned} & \mathbb{E} \left\| \mathbb{K}_\beta^{-1} \left(\int_0^\cdot (G(Z_s^{x_n}) - G(Z_s^x)) ds \right) \right\|_{L^2(0,T,H)}^2 \\ & \leq c_\beta^2 \mathbb{E} \int_0^T \left\| \frac{s^{\beta-\frac{1}{2}}}{\Gamma(\frac{3}{2}-\beta)} \left(\frac{s^{\frac{1}{2}-\beta} (G(Z_s^{x_n}) - G(Z_s^x))}{s^{\beta-\frac{1}{2}}} \right) \right. \\ & \quad \left. + (\beta - \frac{1}{2}) \int_0^s \frac{s^{\frac{1}{2}-\beta} (G(Z_s^{x_n}) - G(Z_s^x)) - r^{\frac{1}{2}-\beta} (G(Z_r^{x_n}) - G(Z_r^x))}{(s-r)^{\beta+\frac{1}{2}}} dr \right\|_H^2 ds. \end{aligned}$$

As in the previous case, by the continuity of G we have $G(Z_s^{x_n}) - G(Z_s^x) \rightarrow 0$ for each $s \in [0, T]$ \mathbb{P} -almost surely. We aim at showing (4.8) by the dominated convergence theorem. By (4.9) we immediately obtain

$$\mathbb{E} \int_0^T \left\| \frac{s^{\beta-\frac{1}{2}}}{\Gamma(\frac{3}{2}-\beta)} \frac{s^{\frac{1}{2}-\beta} (G(Z_s^{x_n}) - G(Z_s^x))}{s^{\beta-\frac{1}{2}}} \right\|_H^2 ds \rightarrow 0, \quad n \rightarrow \infty.$$

Furthermore, we have (for a generic constant c)

$$\begin{aligned} & \mathbb{E} \int_0^T \left\| \int_0^s \frac{s^{\frac{1}{2}-\beta} (G(Z_s^{x_n}) - G(Z_s^x)) - r^{\frac{1}{2}-\beta} (G(Z_r^{x_n}) - G(Z_r^x))}{(s-r)^{\beta+\frac{1}{2}}} dr \right\|_H^2 ds \\ (4.10) \quad & \leq c \mathbb{E} \int_0^T \left(s^{\beta-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-\beta} - r^{\frac{1}{2}-\beta}}{(s-r)^{\beta+\frac{1}{2}}} \|G(Z_r^{x_n}) - G(Z_r^x)\|_H dr \right)^2 ds \\ & \quad + c \mathbb{E} \int_0^T \left(\int_0^s \frac{\|G(Z_s^{x_n}) - G(Z_s^x) - (G(Z_r^{x_n}) - G(Z_r^x))\|_H}{(s-r)^{\beta+\frac{1}{2}}} dr \right)^2 ds. \end{aligned}$$

The first integral on the right-hand side of (4.10) clearly tends to zero by the Hölder continuity condition (4.4) and the inequality

$$\int_0^s \frac{s^{\frac{1}{2}-\beta} - r^{\frac{1}{2}-\beta}}{(s-r)^{\beta+\frac{1}{2}}} dr \leq cs^{1-2\beta}, \quad s \in (0, T).$$

Again by (4.4) and analyticity of the semigroup S we have

$$\frac{\|G(Z_s^{x_n}) - G(Z_s^x) - G(Z_r^{x_n}) + G(Z_r^x)\|_H}{(s-r)^{\beta+\frac{1}{2}}}$$

$$\begin{aligned}
&\leq c \frac{\|S(s)x_n - S(r)x_n\|_H^\alpha + \|S(s)x - S(r)x\|_H^\alpha + \|\tilde{Z}_s - \tilde{Z}_r\|_H^\alpha}{(s-r)^{\beta+\frac{1}{2}}} \\
&\leq c \frac{(s-r)^{\alpha\varepsilon} + (s-r)^{\alpha\delta} \|\tilde{Z}\|_{C^\delta([0,T],H)}}{r^{\alpha\varepsilon} (s-r)^{\beta+\frac{1}{2}}}
\end{aligned}$$

where the generic constant $c \geq 0$ does not depend on $n \in \mathbb{N}$ and $s, r, s > r, s, r \in (0, T)$, and ε is such that $\alpha\varepsilon < 1, \beta + \frac{1}{2} - \alpha\varepsilon < 1$ and $\delta \in (\frac{1}{\alpha}(\beta - \frac{1}{2}), \beta - \lambda)$ (note that this choice is possible by (4.5) and the fact that $\tilde{Z} \in C^\delta([0, T], E)$). This gives us the convergent majorant for the second integral on the right-hand side of (4.10), and (4.8) follows by dominated convergence. \square

Remark 4.4. Note that a sequence (μ_n) of Borel probability measures on E is conditionally sequentially compact in τ if and only if it is equicontinuous, i.e.

$$(4.11) \quad \lim_{k \rightarrow \infty} \sup_n \mu_n(A_k) = 0 \text{ for all } (A_k) \subset \mathcal{B}(E), A_k \searrow \emptyset$$

and therefore $\mu_n \rightarrow \mu$ in τ provided (4.11) and $\mu_n \rightarrow \mu$ in the w^* -topology (that is, weakly in probabilistic sense), cf. [12, Theorem 2.6 and Lemma 3.15].

Now we can complete the proof of Theorem 4.2.

Proof. The equivalence (i) \Leftrightarrow (ii) has been proved in Theorems 2.7 and 2.11. By Proposition 4.1, for each $T > 0$ and $x \in E$ the measures ν_T^x and μ_T^x are equivalent, so we have trivially (ii) \Leftrightarrow (iii). We prove (iv) \Rightarrow (ii) by contradiction. If (ii) is false there exist $x_0, x_1 \in E$ and $T > 0$ such that $\mu_T^{x_0} \perp \mu_T^{x_1}$ (Gaussian measures must be singular unless they are equivalent). By the Feldman-Hájek Theorem, we then have $\mu_T^{x_n} \perp \mu_T^0$ where $x_n = \frac{1}{n}(x_0 - x_1) \rightarrow 0$. By Proposition 4.1 $\nu_T^{x_n} \perp \nu_T^0$ which contradicts (iv). It remains to show (ii) \Rightarrow (iv). The proof is based on the Lemma 4.3 above and follows the idea from [16] (where the proof is given for Markov case).

First, note that (ii) implies that $\mu_T^{x_n} \rightarrow \mu_T^x$ in τ . This easily follows from the Cameron-Martin formula (for the density of $\mu_T^{x_n}$ with respect to μ_T^x , cf. [5] for a similar proof). Also, by [17, Theorem II.21] and Lemma 4.3 we immediately obtain that

$$(4.12) \quad \rho_T(x_n) \rightarrow \rho_T(x) \text{ in } L^1(\Omega)$$

and the sequence of densities $\rho_T(x_n)$ is equiintegrable. By Remark 4.4 it is sufficient to prove

$$(4.13) \quad \sup_n \nu_T^{x_n}(A_k) \rightarrow 0, \quad k \rightarrow \infty$$

for arbitrary $(A_k) \subset \mathcal{B}(E), A_k \searrow \emptyset$, and

$$(4.14) \quad \int \varphi(y) d\nu_T^{x_n}(y) \rightarrow \int \varphi(y) d\nu_T^x(y), \quad n \rightarrow \infty$$

for each $C_b(E)$. We have

$$\begin{aligned}
\sup_n \nu_T^{x_n}(A_k) &= \sup_n \mathbb{E} \mathbf{1}_{A_k}(Z_T^{x_n}) \rho_T(x_n) \\
&\leq K \sup_n \mathbb{E} \mathbf{1}_{A_n}(Z_T^{x_n}) + \sup_n \mathbb{E} \mathbf{1}_{[\rho_T(x_n) > K]} \rho_T(x_n)
\end{aligned}$$

for arbitrary $K > 0$. Now, we have

$$\sup_n \mathbb{E} \mathbf{1}_{[\rho_T(x_n) > K]} \rho_T(x_n) \rightarrow 0, \quad K \rightarrow \infty$$

by equiintegrability of $\rho_T(x_n)$ and

$$\sup_n \mathbb{E} \mathbf{1}_{A_k}(Z_T^{x_n}) \rightarrow 0, \quad k \rightarrow \infty$$

by Remark 4.4 applied to the linear equation, where we use the fact that $\mu_T^{x_n} \rightarrow \mu_T^x$ in τ , and (4.13) follows. Furthermore,

$$\begin{aligned} & \left| \int_E \varphi(y) d\nu_T^{x_n}(y) - \int_E \varphi(y) d\nu_T^x(y) \right| \\ &= |\mathbb{E} \varphi(Z_T^{x_n}) \rho_T(x_n) - \mathbb{E} \varphi(Z_T^x) \rho_T(x)| \\ &\leq \mathbb{E} |(\rho_T(x_n) - \rho_T(x)) \varphi(Z_T^{x_n})| + \mathbb{E} \rho_T(x) |\varphi(Z_T^{x_n}) - \varphi(Z_T^x)| \\ &\leq \sup \|\varphi\| \mathbb{E} |\rho_T(x_n) - \rho_T(x)| + K \mathbb{E} |\varphi(Z_T^{x_n}) - \varphi(Z_T^x)| \\ &\quad + 2 \sup \|\varphi\| \mathbb{E} \mathbf{1}_{[\rho_T(x) > K]} \rho_T(x), \end{aligned}$$

for arbitrary $K > 0$. Clearly, we have $Z_T^{x_n} - Z_T^x = S(T)(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$, hence using (4.12) we obtain (4.14), which concludes the proof of the Theorem. \square

Example 4.5. Consider the 1D stochastic equation of reaction-diffusion type

$$(4.15) \quad \begin{cases} \frac{\partial y}{\partial t}(t, \xi) = \frac{\partial^2 y}{\partial \xi^2}(t, \xi) + f(y, t, \xi) + \eta^\beta(t, \xi), & (t, \xi) \in (0, T) \times (0, 1) \\ y(0, \xi) = x(\xi), & \xi \in (0, 1) \\ y(t, 0) = y(t, 1) = 0, & t \in (0, T), \end{cases}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ and the noise η^β is fractional in time with the Hurst parameter $\beta \in (0, 1)$ and white in space. For the linear case ($f = 0$) it is a particular case of the Example 3.3. The rigorous interpretation of (4.15) is the equation (4.2) where we put $H = E = L^2(0, 1)$, $B = I$, $A = \frac{\partial^2}{\partial \xi^2}$,

$$\text{Dom}(A) = \{\varphi \in E; \varphi, \varphi' \text{ are AC}, \varphi'' \in E, \varphi(0) = \varphi(1) = 0\}$$

and $F : E \rightarrow E$, $(F(x))(\xi) := f(x(\xi))$, is the Nemytskii operator. We need to impose some conditions on f so that the operator $F : E \rightarrow E$ is well-defined and satisfies the assumptions of the present section. To this end we assume that f is continuous and of at most linear growth,

$$(4.16) \quad |f(\xi)| \leq k(1 + |\xi|), \quad \xi \in \mathbb{R},$$

for some $k \geq 0$, and if $\beta > \frac{1}{2}$ we assume in addition that f is also Hölder continuous,

$$(4.17) \quad |f(\xi) - f(\eta)| \leq k|\xi - \eta|^\alpha, \quad \xi, \eta \in \mathbb{R},$$

for some $k \geq 0$ and $\alpha > 0$ satisfying $\alpha > \frac{\beta - \frac{1}{2}}{\beta - \frac{1}{4}}$. Using the fact that $\lambda = \frac{1}{4}$ (cf. Example 3.3) this easily implies that the Hypothesis (G) is satisfied. As we have seen in the Example 3.3, the correspond deterministic system (4.1) is null controllable in this case and we may conclude that for each $T > 0$ the probability laws ν_T^x , $x \in E$, are equivalent and the mapping $x \mapsto \nu_T^x$ is continuous in the topology of pointwise convergence (the strong Feller property holds).

Remark 4.6. (i) In the previous example, the conditions (4.16), (4.17) (which are clearly too strong for usual reaction-diffusion models) may be replaced by

$$\begin{aligned} |f(\xi)| &\leq K_\beta(1 + |\xi|^\rho), \\ |f(\xi) - f(\eta)| \text{ sign}(\xi - \eta) &\leq K_\beta(\xi - \eta), \quad \xi, \eta \in \mathbb{R} \end{aligned}$$

and (if $\beta > \frac{1}{2}$)

$$|f(\xi) - f(\eta)| \leq K_\beta(1 + |\xi|^q + |\eta|^q)|\xi - \eta|^\alpha, \quad \xi, \eta \in \mathbb{R}$$

for some $K \geq 0$, $\rho, q > 0$, and $\alpha \leq 1$ satisfying $\alpha > \frac{\beta - \frac{1}{2}}{\beta - \frac{1}{4}}$. This may be shown in the same way as in the present example, but the proof of the analogue of Lemma 4.3 becomes technically more complicated (and E is no longer Hilbert space).

(ii) Using Example 3.5 we may also consider the semilinear equation with boundary noise

$$\begin{cases} \frac{\partial y}{\partial t}(t, \xi) = \frac{\partial}{\partial \xi} \left(p(\xi) \frac{\partial}{\partial \xi} y \right)(t, \xi) + q(\xi)y(t, \xi), & (t, \xi) \in (0, T) \times (0, 1), \\ y(0, \xi) = x(\xi), & \xi \in (0, 1), \\ \frac{\partial}{\partial \xi} y(t, j) = f_j(y(t, \cdot)) + \sigma_j \dot{B}_j^\beta(t), & t \in (0, T), j \in \{0, 1\}, \end{cases}$$

where $f_j : E \rightarrow \mathbb{R}$, $j \in \{0, 1\}$, satisfy corresponding continuity and growth conditions. Unless both σ_j , $j \in \{0, 1\}$, are zero we again obtain that the strong Feller property holds for the above semilinear stochastic equation.

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FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY IN PRAGUE, SOKOLOVSKA 83,
186 75 PRAGUE 8, CZECH REPUBLIC

E-mail address: maslow@karlin.mff.cuni.cz

DELFT INSTITUTE OF APPLIED MATHEMATICS, DELFT UNIVERSITY OF TECHNOLOGY, P.O. BOX
5031, 2600 GA DELFT, THE NETHERLANDS

E-mail address: J.M.A.M.vanNeerven@TUDelft.nl